HW2: Vorticity and Creeping Flow

To be returned on January 31, 2017

Refs: Falkovich's book and for a more extended reading on these two topics you might consult Chapters 5 and 9 of the book "Fluid Mechanics" by Kundu and Cohen.

I. GENERAL DEFORMATION OF A FLUID ELEMENT

Define the rate of strain tensor as $\varepsilon_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)$, where u_i is the i-th component of the velocity (assumed incompressible $\boldsymbol{\partial}\cdot\boldsymbol{u} = 0$) and ∂_j is the j-th component of the spatial gradient. Consider the velocity at two neighboring points $\boldsymbol{x}$ and $\boldsymbol{x}'$, with the separation $\boldsymbol{s} = \boldsymbol{x}' - \boldsymbol{x}$. First-order Taylor expansion of the velocity yields $\boldsymbol{u}' = \boldsymbol{u} + (\boldsymbol{s}\cdot\boldsymbol{\partial})\,\boldsymbol{u}$.

Prove and use the equality $\frac{1}{2}\left(\boldsymbol{s}\cdot\boldsymbol{\partial}\right)\boldsymbol{u} = \frac{1}{2}\boldsymbol{\omega}\wedge\boldsymbol{s} + \frac{1}{2}\boldsymbol{\partial}\left(\boldsymbol{s}\cdot\boldsymbol{u}\right)$ to demonstrate:

$$\boldsymbol{u}' = \boldsymbol{u} + \frac{1}{2}\boldsymbol{\omega}\wedge\boldsymbol{s} + \frac{1}{2}\boldsymbol{\partial}_s\left(\varepsilon_{ij}s_i s_j\right) . \tag{1}$$

The second term on the r.h.s. represent a local rigid-body rotation, which illustrates the meaning of the vorticity vector $\boldsymbol{\omega}$ as a measure of the local spinning of fluid elements.

As for the last term on the r.h.s., we want to show that it represents a pure straining motion. To that purpose, use incompressibility to show that the rate of strain tensor is traceless. Using this property and diagonalizing the quadratic form $\varepsilon_{ij}s_i s_j$, convince yourself that iso-surfaces of the quadratic form are hyperboloids and the associated gradients $\boldsymbol{\partial}_s\left(\varepsilon_{ij}s_i s_j\right)$ correspond to pure strain, i.e. stretching/squashing in perpendicular directions without any overall rotation.

II. RANKINE VORTEX AS A SIMPLE MODEL FOR TORNADOES

Consider an axisymmetric flow with tangential velocity $\boldsymbol{u} = u(r)\hat{\boldsymbol{e}}_\phi$, where (r,ϕ,z) are cylindrical coordinates. The velocity depends on the radial distance as $u(r) = \Omega a^2/r$ for $r \geq a$ and $u(r) = \Omega r$ for $r \leq a$ where a is a radius of the vortex. Calculate the corresponding vorticity field. Note that the flow is irrotational outside the vortex, i.e. $r > a$.

Real vortices are typically characterized by small vortex cores where the vorticity is concentrated, whilst outside the core the flow is essentially irrotational. The core is not usually circular, nor is the vorticity uniform. In these two respects the Rankine vortex is only a simplified model of real vortices.

Use the Euler equations $\left(\boldsymbol{u}\cdot\boldsymbol{\partial}\right)\boldsymbol{u} = -\boldsymbol{\partial}p/\rho - g\hat{z}$, where g is the gravitational acceleration and ρ is the (constant) density to derive that

$$p(r) = \begin{cases} p_0 - \frac{\rho\Omega^2 a^4}{2r^2} - \rho g z, & r \geq a \\ p_0 - \rho\Omega^2 a^2 + \frac{\rho\Omega^2 r^2}{2} - \rho g z, & r \leq a \end{cases} \tag{2}$$

where p_0 is the atmospheric pressure, i.e. the value at large r and at the surface of the fluid $z = 0$. Conclude that the pressure at $z = 0$ in the center of the vortex is lower than the atmospheric pressure by an amount $\rho\Omega^2 a^2$. The depression in the core of a tornado is a major cause of its destructive effects.

Deduce that the free surface of the liquid at $r = 0$ is at a depth $\Omega^2 a^2/g$ below the surface at infinity (hence the dimples when a cup of liquid is stirred by a rotating spoon).

III. SLOW SWIMMING OF A THIN FLEXIBLE SHEET

Consider a thin extensible sheet that flexes itself in such a way that its coordinates $(x_s, y_s) = (x, a\sin(kx - \omega t))$, i.e. it oscillates in the vertical direction and a wave travels with velocity $c = \omega/k$ to the right. Such a motion is not time-reversible and we want to show that it results in a steady flow component U of the fluid above the sheet. The velocity can be calculated explicitly in the limit $\epsilon = ka \ll 1$ and $U = \epsilon^2 c/2$.

Introduce the stream function ψ such that the two components of the velocity $(u, v) = (\partial_y\psi, -\partial_x\psi)$. Consider the Stokes limit of small Reynolds numbers and show that the bi-Laplacian of ψ vanishes. Write down the boundary conditions dictated by the motion of the sheet $(x_s(t), y_s(t))$. Reduce the equations to a non-dimensional form by appropriate rescalings and assume $\epsilon = ka$ small, i.e. deviations of the height of the free surface from $y = 0$ are small.

We shall seek a perturbative solution $\psi = \psi_0 + \epsilon\psi_1 + \ldots$. Expand in ϵ the bi-Laplacian equation and the boundary conditions at the surface of the flexible sheet, and write down the corresponding equation and boundary conditions up to first order. Imposing the boundary condition that the flow stay finite as $y \to +\infty$, find the expression for ψ_0 and ψ_1 and verify that the latter tends to a constant as $y \to \infty$. Show that the constant coincides (in the original variables) with U.

IV. FLOW IN A THIN FILM

Consider a viscous fluid in steady flow between two rigid fixed boundaries at $z = 0$ and $z = h$. Let U and L be the typical horizontal velocity and length scale. Thin films are characterized by $h \ll L$ so that vertical gradients are stronger than horizontal ones, e.g. the Laplacian $\Delta \sim \partial^2/\partial z^2$. Estimate the order of magnitude of the various terms in the Navier-Stokes equation and show that inertia can be neglected if $UL/\nu \times (h/L)^2 \ll 1$, i.e. the relevant Reynolds number is reduced by the small factor $(h/L)^2$. For geometric reasons one also expects that the vertical component w of the velocity is reduced by a factor h/L wrt to the horizontal ones (u, v). Comparing the Stokes equations for (u, v, w) conclude that pressure is a function of x and y only, at the dominant order. Use this fact to integrate the equations for u and v and impose the no-slip boundary conditions at $z = 0$ and $z = h$ to find the final expression of u and v. The correct solution satisfies the property that the ratio between u and v is independent of z. Calculate the circulation $\Gamma = \int_{\mathcal{C}} u\, dx + v\, dy$ over any closed contour $\mathcal{C}$ in the $x - y$ plane (at any height z) and show that it vanishes.